

# Spherically symmetric equilibria for self-gravitating kinetic or fluid models in the non-relativistic and relativistic case—A simple proof for finite extension

Tobias Ramming, Gerhard Rein  
 Fakultät für Mathematik, Physik und Informatik  
 Universität Bayreuth  
 D-95440 Bayreuth, Germany  
 email: tobias.ramming@uni-bayreuth.de  
 gerhard.rein@uni-bayreuth.de

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## Abstract

We consider a self-gravitating collisionless gas as described by the Vlasov-Poisson or Einstein-Vlasov system or a self-gravitating fluid ball as described by the Euler-Poisson or Einstein-Euler system. We give a simple proof for the finite extension of spherically symmetric equilibria, which covers all these models simultaneously. In the Vlasov case the equilibria are characterized by a local growth condition on the microscopic equation of state, i.e., on the dependence of the particle distribution on the particle energy, at the cut-off energy  $E_0$ , and in the Euler case by the corresponding growth condition on the equation of state  $p = P(\rho)$  at  $\rho = 0$ . These purely local conditions are slight generalizations to known such conditions.

## 1 Introduction

In astrophysics, matter distributions which interact by gravity arise on many different scales. While the Euler-Poisson system can be used as a simple model for a single star, a large ensemble of stars such as a galaxy or globular cluster where collisions among the stars are sufficiently rare to be neglected is typically modeled by the Vlasov-Poisson system; both systems are non-relativistic and possess relativistic counterparts. We refer to [3, 4] for astrophysical background of these systems. In a well known approach to

constructing corresponding equilibrium solutions, the system under investigation is—by a suitable ansatz—reduced to a semi-linear elliptic equation for the potential or its relativistic analogue. The crucial question then is, under which assumptions on the ansatz the resulting steady state has finite mass and compact support, since only such states are of possible interest from a physics point of view. In the present paper we give a simple proof for these finiteness properties which works for all the indicated models simultaneously and covers (and slightly extends) all those cases known from the literature where the assumption is purely local at the cut-off energy or at  $\rho = 0$  respectively.

In order to be more precise we first consider the models where matter is described as a collisionless gas; for the necessary details of what we outline below we refer to the next section. In the non-relativistic and time independent case the ensemble of particles (stars) is described by its density on phase space,  $f = f(x, v) \geq 0$ ,  $x, v \in \mathbb{R}^3$ , which obeys the Vlasov-Poisson system

$$v \cdot \nabla_x f - \nabla U \cdot \nabla_v f = 0, \quad (1.1)$$

$$\Delta U = 4\pi\rho, \quad \lim_{|x| \rightarrow \infty} U(x) = 0, \quad (1.2)$$

$$\rho(x) = \int f(x, v) dv. \quad (1.3)$$

Here  $U = U(x)$  denotes the gravitational potential and  $\rho$  the spatial mass density induced by  $f$ ; we assume that all the particles have the same mass which we normalize to unity. Clearly, the particle energy

$$E = E(x, v) = \frac{1}{2}|v|^2 + U(x) \quad (1.4)$$

satisfies the Vlasov equation (1.1), and hence any function of the form

$$f = \phi(E) \quad (1.5)$$

with a suitable, prescribed function  $\phi$  does as well. The time independent Vlasov-Poisson system thus is reduced to the semi-linear Poisson equation

$$\Delta U = 4\pi \int \phi \left( \frac{1}{2}|v|^2 + U \right) dv, \quad \lim_{|x| \rightarrow \infty} U(x) = 0, \quad (1.6)$$

and the question is under what conditions on  $\phi$  the latter equation has solutions and whether the resulting steady states have finite mass and compact support. A necessary condition for the latter is that  $\phi(E) = 0$  for  $E > E_0$  where  $E_0$  is a suitable cut-off energy, cf. [25, Thm. 2.1].

If instead we describe the matter as an ideal, compressible fluid, then all that remains of the Euler equations in the static, time independent case is the equation

$$\nabla p + \rho \nabla U = 0, \quad (1.7)$$

where the pressure  $p$  depends on the mass density  $\rho$  via an equation of state

$$p = P(\rho), \quad (1.8)$$

and the gravitational potential obeys the Poisson equation (1.2). If  $P$  is strictly increasing on  $[0, \infty[$  then (1.7) and (1.8) can be used to express  $\rho$  as a function of  $U$ , and again the system is reduced to a semi-linear Poisson equation.

The results in [6] imply that solutions to these semi-linear Poisson problems which lead to finite mass and compact support must be spherically symmetric. Hence we do not loose any relevant equilibria if we make this assumption from the start. Under this assumption the characteristic flow of the Vlasov equation has the additional invariant

$$L := |x \times v|^2, \quad (1.9)$$

the modulus of angular momentum squared. We generalize the ansatz (1.5) to

$$f = \phi(E) L^l \quad (1.10)$$

which allows for a certain anisotropy in the Vlasov case; here  $l > -1/2$ .

Suppose now that we wish to describe the analogous physical systems in a relativistic set-up. On the kinetic level we can consider the so-called relativistic Vlasov-Poisson system, where the Vlasov equation is changed to

$$\frac{v}{\sqrt{1 + |v|^2}} \cdot \nabla_x f - \nabla U \cdot \nabla_v f = 0, \quad (1.11)$$

while the Poisson equation (1.2) and (1.3) remain unchanged; like all other physical constants the speed of light is normalized to unity. The particle energy is redefined as

$$E = E(x, v) = \sqrt{1 + |v|^2} + U(x), \quad (1.12)$$

and the same reduction procedure as outlined above applies. For the static Euler equations the velocity field is identically zero, and hence there is no difference between the Euler-Poisson and the relativistic Euler-Poisson systems here.

The relativistic Vlasov-Poisson system is neither Galilei nor Lorentz invariant and is only included here to show that our simple proof covers all the models of this type. The genuinely relativistic case has to be modeled in the context of general relativity. Assuming spherical symmetry we use Schwarzschild coordinates and write the metric in the form

$$ds^2 = -e^{2\mu} dt^2 + e^{2\lambda} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

where Schwarzschild time  $t$  coincides with the proper time of an observer who is at rest at spatial infinity,  $r \geq 0$  is the area radius, and the polar angles  $\theta \in [0, \pi]$  and  $\varphi \in [0, 2\pi]$  coordinatize the orbits of symmetry. The static Einstein-Vlasov system takes the form

$$\frac{v}{\sqrt{1+|v|^2}} \cdot \nabla_x f - \sqrt{1+|v|^2} \mu' \frac{x}{r} \cdot \nabla_v f = 0, \quad (1.13)$$

$$e^{-2\lambda} (2r\lambda' - 1) + 1 = 8\pi r^2 \rho, \quad (1.14)$$

$$e^{-2\lambda} (2r\mu' + 1) - 1 = 8\pi r^2 p, \quad (1.15)$$

$$\rho(r) = \rho(x) = \int \sqrt{1+|v|^2} f(x, v) dv, \quad (1.16)$$

$$p(r) = p(x) = \int \left( \frac{x \cdot v}{r} \right)^2 f(x, v) \frac{dv}{\sqrt{1+|v|^2}}. \quad (1.17)$$

Here  $x = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)$  so that  $r = |x|$  is the Euclidean norm of  $x \in \mathbb{R}^3$ ,  $\cdot$  denotes the Euclidean scalar product on  $\mathbb{R}^3$ ,  $f$  is spherically symmetric, i.e.,  $f(x, v) = f(Ax, Av)$  for all  $A \in \text{SO}(3)$ , and  $'$  denotes the derivative with respect to  $r$ . As to the choice of the momentum variable  $v \in \mathbb{R}^3$  which leads to the above form of the system we refer to [18, 23]. As boundary conditions we require asymptotic flatness, i.e.,

$$\lim_{r \rightarrow \infty} \mu(r) = \lim_{r \rightarrow \infty} \lambda(r) = 0, \quad (1.18)$$

and a regular center, i.e.,

$$\lambda(0) = 0. \quad (1.19)$$

For the static Einstein-Vlasov system the particle energy takes the form

$$E = E(x, v) = e^{\mu(x)} \sqrt{1+|v|^2}, \quad (1.20)$$

and an ansatz of the form (1.10) again satisfies the corresponding Vlasov equation. With this ansatz the quantities  $\rho$  and  $p$  defined in (1.16) and

(1.17) become functions of  $\mu$ . Moreover,  $e^{-2\lambda}$  can, via (1.14) and (1.19), be expressed in terms of  $\rho$ , and the whole system is reduced to a single equation for  $\mu$  which arises from (1.15).

To complete the set of models which we consider we turn to the Einstein-Euler system. Since we consider the static case, the Euler equations reduce to the single equation

$$\nabla p + (p + \rho) \nabla \mu = 0, \quad (1.21)$$

where the pressure  $p$  depends on  $\rho$  via an equation of state like (1.8). The field equations (1.14), (1.15) together with the boundary conditions (1.18), (1.19) remain unchanged. As in the case of the Euler-Poisson system  $\rho$  becomes a function of  $\mu$ , and the system is reduced to a single equation for  $\mu$ .

Up to technical requirements we make the following assumptions. In the kinetic case,

$$\phi(E) \geq c(E_0 - E)^k$$

for  $E < E_0$  close to the cut-off energy  $E_0$ , where  $c > 0$  and  $k < l + 3/2$ . For the fluid case we require that  $P(\rho)$  is strictly increasing for  $\rho > 0$  with

$$P'(\rho) \leq c\rho^{1/n}$$

for  $\rho > 0$  and small, where  $0 < n < 3$ . In passing we remark that if one computes the pressure induced by the ansatz (1.10) in the isotropic case  $l = 0$  then it can be written as a function of  $\rho$  which satisfies the fluid case assumption with  $n = k + 3/2$ , and the restrictions on the growth rates fit.

We compare our result with known results from the literature. The classical example in the context of the Vlasov-Poisson system are the polytropic models where

$$f(x, v) = (E_0 - E)_+^k L^l; \quad (1.22)$$

the subscript  $+$  denotes the positive part. The resulting semi-linear Poisson equation is the Lane-Emden-Fowler equation. Based on the analysis in [27] the corresponding steady states were analyzed in [2]. Here  $k, l > -1$  with  $k + l + 3/2 \geq 0$ . Compactly supported steady states are obtained for  $k < 3l + 7/2$ , for  $k = 3l + 7/2$  the mass is still finite but the support is  $\mathbb{R}^3$ , and for  $k > 3l + 7/2$  the mass becomes infinite. In [17, 24] extensions of these and related results were given for the Einstein-Vlasov system. In [25] it was shown both for the Vlasov-Poisson and the Einstein-Vlasov systems that a sufficient condition for a compact support is that the ansatz is of the form (1.22) asymptotically for  $E \rightarrow E_0$ , but this purely local condition is sufficient only if  $k < l + 3/2$ . This result was motivated by and relied on a corresponding

analysis for the Einstein-Euler system [14]. In [25] a list of examples from the astrophysics literature is given which are covered by such a purely local condition, and it is also demonstrated by a suitable counterexample that such purely local methods fail for  $k > l + 3/2$ . The analysis in the present paper relies on a purely local characterization of the (microscopic) equation of state as well and is subject to the same restriction. Results which are not subject to this restriction have been investigated by quite sophisticated dynamical systems methods in [5, 9, 10, 11]. Using a global characterization of the ansatz function  $\phi$  these results cover polytropes for the full range of exponents mentioned above under a size restriction on the initial data. Our analysis is much closer to the ones in [14, 25], our conditions are less restrictive in that only an estimate and not an asymptotic behavior is needed at the cut-off, but more important from our point of view is that our proof is transparent and short—cf. Section 3—, and it applies to all the different systems specified above.

There is by now a rich literature on the stability of steady states of the Vlasov-Poisson system, cf. [7, 8, 13, 22] and the references there. It is interesting that the character of the stability analysis changes at the threshold  $k = 3/2$ — $k$  is again a growth rate for the ansatz function and  $l = 0$  here—in the sense that below this threshold one can use a reduction procedure which gives a stability result simultaneously for the Vlasov-Poisson and Euler-Poisson systems while such an approach does not work above this threshold [20, 21, 22]. We refer to [12] for a complementary instability result in the Euler-Poisson case.

The paper proceeds as follows. In the next section we show in more detail how in the static case the models we consider can be reduced to a single equation for a suitably defined function  $y$ , related to either  $U$  or  $\mu$ . The arguments there are known, but we need to put them into a common framework. The steady state under consideration has compact support if and only if the function  $y$ , which starts with a positive value at the origin and is decreasing, has a zero. It turns out that in all the cases considered,  $y$  satisfies an inequality of the form

$$y'(r) \leq -\frac{m(r)}{r^2}, \quad (1.23)$$

where the mass function  $m$  is defined in terms of  $\rho$  by

$$m(r) = 4\pi \int_0^r s^2 \rho(s) ds,$$

and  $\rho(r) = r^{2l}g(y(r))$  is given in terms of  $y$ . In Section 3 we prove that under a condition on the behavior of  $g$  at  $y = 0$ , all functions which satisfy

(1.23) have a zero. This rests on two simple observations. Firstly, the mass function is increasing since the mass-energy density  $\rho$  is non-negative, and secondly, the latter function is in the present context always decreasing, up to the possible anisotropy factor  $r^{2l}$  in the Vlasov case. In the last section we translate the general condition from Section 3 into a condition on the microscopic equation of state  $\phi$  or the macroscopic equation of state  $P$  respectively and obtain the compact support property for all the models considered above.

## 2 The basic set-up

In this section we discuss in more detail how the analysis of steady states for the systems which were introduced above can be reduced to that of a suitable master equation for the potential or a related quantity.

### 2.1 Kinetic models

#### 2.1.1 The Vlasov-Poisson system

In the ansatz (1.10) a cut-off energy  $E_0$  has to be specified which is the value of the potential at the boundary of the support of the matter. On the other hand we have the standard boundary condition in (1.2) at infinity, and due to spherical symmetry it seems natural to parametrize for a fixed ansatz function  $\phi$  the solutions by prescribing the value  $U(0)$  of the potential at the center. Since this is one free parameter respectively one condition too many it is natural to slightly modify the ansatz (1.10). We prescribe a function  $\Phi$  and make the ansatz

$$f(x, v) = \Phi(E_0 - E)L^l \quad (2.1)$$

with  $l > -1/2$ , and we look for  $y = E_0 - U$  with a prescribed value at the origin,  $y(0) = \mathring{y} > 0$ . Once a solution  $y$  with a zero is found we define  $E_0 := \lim_{r \rightarrow \infty} y(r)$  and  $U := E_0 - y$ . In this way the cut-off energy  $E_0$  is eliminated as a free parameter and becomes part of the solution. The following technical assumption on  $\Phi$  is required for the reduction procedure, but it does in general not guarantee the compact support of the resulting steady states.

**Assumptions on  $\Phi$ .**  $\Phi : \mathbb{R} \rightarrow [0, \infty[$  is measurable,  $\Phi(\eta) = 0$  for  $\eta < 0$ , and  $\Phi > 0$  a. e. on some interval  $[0, \eta_1]$  with  $\eta_1 > 0$ . Moreover, there exists  $\kappa > -1$  such that for every compact set  $K \subset \mathbb{R}$  there exists a constant  $C > 0$  such that

$$\Phi(\eta) \leq C\eta^\kappa, \quad \eta \in K.$$

If we substitute the ansatz (2.1) into the definition (1.3) of  $\rho$  we find after a short computation that for  $U(r) < E_0$ ,

$$\begin{aligned}\rho(r) &= c_l r^{2l} \int_{U(r)}^{E_0} \Phi(E_0 - E) (E - U(r))^{l+1/2} dE \\ &= c_l r^{2l} \int_0^{E_0 - U(r)} \Phi(\eta) (E_0 - U(r) - \eta)^{l+1/2} d\eta\end{aligned}$$

and  $\rho(r) = 0$  if  $U(r) \geq E_0$ . Here

$$c_l := 2^{l+3/2} \pi \int_0^1 \frac{s^l}{\sqrt{1-s}} ds.$$

Hence in terms of  $y := E_0 - U$  we find that

$$\rho(r) = r^{2l} g(y(r)) \quad (2.2)$$

where

$$g(y) := \begin{cases} c_l \int_0^y \Phi(\eta) (y - \eta)^{l+1/2} d\eta, & y > 0, \\ 0, & y \leq 0. \end{cases} \quad (2.3)$$

Under the above assumptions on  $\Phi$  it follows by Lebesgue's dominated convergence theorem that  $g \in C(\mathbb{R}) \cap C^1([0, \infty[)$  with

$$g'(y) = (l + 1/2) c_l \int_0^y \Phi(\eta) (y - \eta)^{l-1/2} d\eta, \quad y > 0,$$

and  $g \in C^1(\mathbb{R})$  if  $\kappa + l + 1/2 > 0$ . Due to spherical symmetry the semi-linear Poisson equation (1.6) can in terms of  $y$  be written as

$$\frac{1}{r^2} (r^2 y')' = -4\pi r^{2l} g(y). \quad (2.4)$$

In terms of the Cartesian variables we want potentials  $U \in C^2(\mathbb{R}^3)$  i.e.,  $y \in C^2(\mathbb{R}^3)$ . Hence we require that  $y'(0) = 0$ , integrate (2.4) once and have to solve the equation

$$y'(r) = -\frac{m(r)}{r^2} \quad (2.5)$$

where

$$m(r) = m(r, y) = 4\pi \int_0^r s^{2l+2} g(y(s)) ds. \quad (2.6)$$

For any  $\mathring{y} > 0$  the equation (2.5) has a unique solution  $y \in C^1([0, \infty[)$  with  $y(0) = \mathring{y}$ , and we briefly review the proof. Firstly, a standard contraction



argument shows that there is a unique, local solution on some short interval  $[0, \delta[$ . This solution extends uniquely to a maximal interval  $[0, r_{\max}[$  where by monotonicity,  $0 < y(r) < \overset{\circ}{y}$ . If  $r_{\max} = \infty$ , we are done, if not, then necessarily  $y(r_{\max}) = 0$ , and again by monotonicity,  $y$  uniquely extends to the right via

$$y'(r) = -\frac{m(r_{\max})}{r^2}, \quad r > r_{\max}.$$

In addition,  $y'(0) = 0$ , and the regularity of  $g$  implies that  $y \in C^2(\mathbb{R}^3)$  as desired.

In the next section we specify a condition on  $g$  which guarantees that the solution  $y$  has a zero at some finite radius  $R$ . In the last section we translate that condition into one on the ansatz function  $\Phi$ , and from  $y$  and  $\Phi$  we then generate a steady state of the Vlasov-Poisson system which in space is supported on the ball with radius  $R$  centered at the origin.

### 2.1.2 The relativistic Vlasov-Poisson system

We make the same ansatz (2.1) as for the Vlasov-Poisson system, with a function  $\Phi$  which has the same properties as in 2.1.1, but the particle energy is now defined by (1.12). We again reduce the full system to the equation (2.5), but now for  $y = E_0 - U - 1$ ; note that by (1.12),  $E \geq 1 + U(r)$ . The mass function  $m$  is defined as in (2.6), but in the relation (2.16) we obtain a different form for the function  $g$ :

$$g(y) := \begin{cases} c_l \int_0^y \Phi(\eta) (1 + y - \eta) ((1 + y - \eta)^2 - 1)^{l+1/2} d\eta, & y > 0, \\ 0, & y \leq 0, \end{cases} \quad (2.7)$$

where

$$c_l := 2\pi \int_0^1 \frac{s^l}{\sqrt{1-s}} ds.$$

The function  $g$  looks more complicated now, but it has the same properties which were stated in the Vlasov-Poisson case, and we arrive at the same type of set-up as in 2.1.1.

### 2.1.3 The Einstein-Vlasov system

First we observe that the unique solution to the field equation (1.14) which satisfies the boundary condition (1.19) is given by

$$e^{-2\lambda(r)} = 1 - \frac{2m(r)}{r}, \quad (2.8)$$

where the mass function  $m$  is defined as above. This relation defines  $\lambda$  only as long as the right hand side is positive, a restriction which is due to the fact that Schwarzschild coordinates cannot cover regions of spacetime which contain a trapped surface. If we eliminate  $\lambda$  via (2.8) and observe that the particle energy (1.20) depends only on  $\mu$ , the static Einstein-Vlasov system is reduced to a single equation for  $\mu$ , namely to (1.15).

In order to arrive at a master equation for a suitable quantity  $y$  which is qualitatively of the same form as before we need to adapt the ansatz to the fact that the particle energy (1.20) is no longer the sum of a kinetic and a potential part. Hence we make the ansatz that

$$f(x, v) = \Phi \left( 1 - \frac{E}{E_0} \right) L^l, \quad (2.9)$$

where  $\Phi$  has the properties stated in 2.1.1. We define  $y := \ln E_0 - \mu$  so that  $e^\mu = E_0/e^y$ ; notice that the particle energy (1.20) is always positive so we require that  $E_0 > 0$ . If we substitute the above ansatz into the definitions (1.16) and (1.17) we obtain the relations

$$\rho(r) = r^{2l} g(y(r)), \quad p(r) = r^{2l} h(y(r)), \quad (2.10)$$

where

$$g(y) := \begin{cases} c_l e^{3y} \int_0^{1-e^{-y}} \Phi(\eta) (1-\eta)^2 (e^{2y}(1-\eta)^2 - 1)^{l+1/2} d\eta, & y > 0, \\ 0, & y \leq 0, \end{cases} \quad (2.11)$$

with  $c_l$  as in 2.1.2, and

$$h(y) := \begin{cases} d_l e^y \int_0^{1-e^{-y}} \Phi(\eta) (e^{2y}(1-\eta)^2 - 1)^{l+3/2} d\eta, & y > 0, \\ 0, & y \leq 0, \end{cases} \quad (2.12)$$

with

$$d_l := 2\pi \int_0^1 s^l \sqrt{1-s} ds.$$

The functions  $g$  and  $h$  have the same regularity properties as the function  $g$  in 2.1.1, cf. [25, Lemma 2.2], and the static Einstein-Vlasov system is reduced to the equation

$$y'(r) = -\frac{1}{1-2m(r)/r} \left( \frac{m(r)}{r^2} + 4\pi r p(r) \right), \quad (2.13)$$

where the mass function  $m$  is defined in terms of  $\rho$  as before and  $\rho$  and  $p$  are given in terms of  $y$  by (2.10). For any  $\hat{y} > 0$  there exists a unique

solution  $y \in C^1([0, \infty[)$  of (2.13) with  $y(0) = \mathring{y}$  which due to the issue of the positivity of the denominator is less easy to see, cf. [17, 24].

Once a solution to (2.13) is obtained, we define  $y_\infty := \lim_{r \rightarrow \infty} y(r)$ ,  $E_0 := e^{y_\infty}$ , and  $\mu = \ln E_0 - y$ . Together with the ansatz (2.9) this yields a steady state with all the desired properties, provided  $y$  has a zero. It turns out that in order to show the latter not the full information of (2.13) is needed, but only the following inequality which immediately follows from that equation:

$$y'(r) \leq -\frac{m(r)}{r^2}. \quad (2.14)$$

The difference between the Newtonian, special relativistic, or general relativistic cases is then reflected only in the definition of the function  $g$ , and only an estimate for  $g$  at  $y = 0$  which holds in all three cases is needed to guarantee a zero for  $y(r)$ .

## 2.2 Fluid models

### 2.2.1 The Euler-Poisson system

We use the static Euler equation (1.7) together with the equation of state (1.8) in order to express  $\rho$  in terms of  $U$ .

**Assumptions on  $P$ .** Let  $P \in C^1([0, \infty[)$  be such that  $P' > 0$  on  $]0, \infty[$ , and

$$\int_0^1 \frac{P'(s)}{s} ds < \infty.$$

We define

$$Q(\rho) := \int_0^\rho \frac{P'(s)}{s} ds, \quad \rho \geq 0,$$

so that  $Q \in C([0, \infty[) \cap C^1(]0, \infty[)$  with  $Q(0) = 0$  and  $Q'(\rho) = P'(\rho)/\rho$  for  $\rho > 0$ . When written in the radial variable  $r$ , (1.7) reads

$$P'(\rho) \rho' + \rho U' = 0. \quad (2.15)$$

If we divide by  $\rho$ , integrate with respect to  $r$  and apply a change of variables it turns out that the pair  $(\rho, U)$  satisfies (2.15)—at least where  $\rho(r) > 0$ —, provided

$$Q(\rho(r)) = c - U(r), \quad r \geq 0, \quad (2.16)$$

with some integration constant  $c$  which like the cut-off energy  $E_0$  above is the value of the potential at the boundary of the matter support. Let

$$y_{\max} := \int_0^\infty \frac{P'(s)}{s} ds = \lim_{\rho \rightarrow \infty} Q(\rho) \in ]0, \infty].$$

Then  $Q : [0, \infty[ \rightarrow [0, y_{\max}[$  is one-to-one and onto, and we define

$$g(y) := \begin{cases} Q^{-1}(y), & 0 < y < y_{\max}, \\ 0, & y \leq 0. \end{cases} \quad (2.17)$$

Then  $g \in C([-\infty, y_{\max}[) \cap C^1([0, y_{\max}[)$ , and writing  $y = c - U$  we invert the relation (2.16) to read as in (2.2) with  $l = 0$  there. Hence the static Euler-Poisson system is reduced to the same equation (2.5) with mass function defined by (2.6) with  $l = 0$  and with  $g$  defined by (2.17) instead of by (2.3). Hence we are in exactly the same situation as in the Vlasov-Poisson case in that the crucial question is whether  $y$  has a zero  $R$ .

### 2.2.2 The Einstein-Euler system

Similarly to 2.2.1 we use the static Euler equation (1.21) together with the equation of state (1.8) in order to express  $\rho$  in terms of  $\mu$ .

**Assumptions on  $P$ .** Let  $P \in C^1([0, \infty[)$ ,  $P \geq 0$  be such that  $P' > 0$  on  $]0, \infty[$ , and

$$\int_0^1 \frac{P'(s)}{s + P(s)} ds < \infty.$$

We define

$$Q(\rho) := \int_0^\rho \frac{P'(s)}{s + P(s)} ds, \quad \rho \geq 0,$$

so that  $Q \in C([0, \infty[) \cap C^1([0, \infty[)$  with  $Q(0) = 0$  and  $Q'(\rho) = P'(\rho)/(\rho + P(\rho))$  for  $\rho > 0$ . We rewrite (1.21) in the radial variable  $r$ ,

$$P'(\rho) \rho' + (\rho + P(\rho)) \mu' = 0. \quad (2.18)$$

If we divide by  $\rho + P(\rho)$ , integrate with respect to  $r$  and apply a change of variables we see that the pair  $(\rho, \mu)$  satisfies (2.18)—at least where  $\rho(r) > 0$ —, provided

$$Q(\rho(r)) = c - \mu(r), \quad r \geq 0, \quad (2.19)$$

with some integration constant  $c$ . Let

$$y_{\max} := \int_0^\infty \frac{P'(s)}{s + P(s)} ds = \lim_{\rho \rightarrow \infty} Q(\rho) \in ]0, \infty].$$

Then as before  $Q : [0, \infty[ \rightarrow [0, y_{\max}[$  is one-to-one and onto, and we define  $g$  by (2.17) which has the same regularity properties as before. Writing  $y = c - \mu$  we can invert the relation (2.19) to read as in (2.2) with  $l = 0$  there. Hence the static Einstein-Euler system is reduced to the same equation (2.13) with mass function defined by (2.6) with  $l = 0$ . The issue again is whether  $y$  has a zero, and this will be determined by the inequality (2.14).

### 3 The compact-support-Lemma

The key to the compact support property for all the models discussed above is the following result.

**Lemma 3.1** *Let  $y \in C^1([0, \infty[)$  with  $y(0) = \mathring{y} \in ]0, y_{\max}[$  satisfy the estimate*

$$y'(r) \leq -\frac{m(r)}{r^2} \text{ on } [0, \infty[,$$

where

$$m(r) = m(r, y) := 4\pi \int_0^r s^{2+2l} g(y(s)) ds,$$

$g \in C([-\infty, y_{\max}[)$  is increasing with  $g(y) = 0$  for  $y \leq 0$  and  $g(y) > 0$  for  $y > 0$ , and  $l > -1/2$ . Let  $g$  satisfy the estimate

$$g(y) \geq c y^{n+l} \text{ for } 0 < y < y^*$$

with parameters  $c > 0$ ,  $y^* > 0$ , and  $0 < n < 3 + l$ . Then the function  $y$  has a unique zero.

**Proof.** Since  $y$  is decreasing, the limit  $y_\infty := \lim_{r \rightarrow \infty} y(r) \in [-\infty, \infty[$  exists, and we need to show that  $y_\infty < 0$ . This will imply the existence of a zero of  $y$  which will be unique by the strict monotonicity of this function. Assume that  $y_\infty > 0$ . Then  $y(r) \geq y_\infty$  on  $[0, \infty[$ , and by the monotonicity of  $g$ ,

$$m(r) \geq 4\pi g(y_\infty) \int_0^r s^{2+2l} ds = \frac{4\pi}{2l+3} g(y_\infty) r^{2l+3}.$$

If we put this into the estimate for  $y'$  and integrate we obtain the contradiction

$$y(r) \leq \mathring{y} - Cr^{2+2l} \rightarrow -\infty \text{ as } r \rightarrow \infty;$$

$C$  denotes a positive constant which may depend on all the parameters, may change from line to line, but never depends on  $r$ .

The argument so far is standard and well known, and the crucial task is to derive a contradiction from the remaining possibility that  $y_\infty = 0$ . Firstly, we observe that  $m$  is increasing in  $r$  and positive for  $r > 0$ . Hence

$$m(r) \geq m(1) =: m_1 > 0 \text{ for } r \geq 1, \quad (3.1)$$

and

$$y(r) = - \int_r^\infty y'(s) ds \geq m_1 \int_r^\infty \frac{ds}{s^2} = \frac{m_1}{r} \text{ for } r \geq 1. \quad (3.2)$$

Secondly, since  $g$  is increasing and  $y$  decreasing,

$$m(r) \geq 4\pi g(y(r)) \int_0^r s^{2+2l} ds = \frac{4\pi}{2l+3} r^{2l+3} g(y(r)). \quad (3.3)$$

Hence

$$y'(r) \leq -\frac{4\pi}{2l+3} r^{2l+1} g(y(r)), \quad r > 0.$$

By a simple change of variables this implies that for all  $r > 0$ ,

$$\int_{y(r)}^{\bar{y}} \frac{d\eta}{g(\eta)} = - \int_0^r \frac{y'(s)}{g(y(s))} ds \geq \frac{4\pi}{2l+3} \int_0^r s^{2l+1} ds = \frac{4\pi}{(2l+3)(2l+2)} r^{2l+2}.$$

Now we take  $r > 0$  sufficiently large so that  $0 < y(r) < y^*$ ; recall that by assumption,  $y(r) \rightarrow 0$  as  $r \rightarrow \infty$ . Then by the growth assumption on  $g$ ,

$$C_1 r^{2l+2} \leq \int_{y(r)}^{y^*} \frac{d\eta}{g(\eta)} + C_2 \leq \frac{1}{c} \int_{y(r)}^{y^*} \frac{d\eta}{\eta^{n+l}} + C_2.$$

We estimate the left hand side from below using (3.2), multiply the resulting estimate by  $y(r)^{2l+2}$  and compute the integral where we need to distinguish the cases  $n+l \neq 1$  and  $n+l = 1$ . In the former case we find that

$$C_1 \leq \frac{1}{c(1-l-n)} \left( (y^*)^{1-l-n} - y(r)^{1-l-n} \right) y(r)^{2l+2} + C_2 y(r)^{2l+2},$$

in the latter

$$C_1 \leq \frac{1}{c} \ln \left( \frac{y^*}{y(r)} \right) y(r)^{2l+2} + C_2 y(r)^{2l+2},$$

which holds for  $r$  sufficiently large with positive constants  $C_1, C_2$ . Since  $2l+2 > 0$  and  $l+3-n > 0$  and since by assumption  $y_\infty = 0$ , the right hand side goes to zero as  $r$  goes to infinity which is the desired contradiction.  $\square$

**Remark.** The two estimates (3.1) and (3.3) on which the above proof rests are quite obvious from a physics point of view. The mass function  $m(r)$  is increasing in  $r$  since the mass-energy density  $\rho$  is non-negative, and this yields (3.1), and  $\rho$  is, up to a possible anisotropy factor  $r^{2l}$  in the Vlasov case, a decreasing function, which yields (3.3).

## 4 Application to the various models

In this section we apply Lemma 3.1 to the various models. We have to check what type of ansatz function  $\Phi$  or equation of state function  $P$  leads to a relation between the mass-energy density  $\rho$  and the function  $y$  with a functional dependence  $g$  which satisfies the growth condition in that lemma.

## 4.1 Kinetic models

Let  $\Phi$  satisfy the assumptions stated in 2.1.1. In addition let

$$\Phi(\eta) \geq c\eta^k \text{ for } \eta \in ]0, \eta_0[ \quad (4.1)$$

for some parameters  $c > 0$ ,  $\eta_0 > 0$ , and  $-1 < k < l + 3/2$ .

### 4.1.1 The Vlasov-Poisson system

For  $0 < y < \eta_0$  the function  $g$  defined by (2.3) satisfies the following estimate;  $C > 0$  denotes a constant which can change from line to line and depends only on the parameters above:

$$\begin{aligned} g(y) &\geq C \int_0^y \eta^k (y - \eta)^{l+1/2} d\eta = Cy^{l+1/2} \int_0^y \eta^k (1 - \eta/y)^{l+1/2} d\eta \\ &= Cy^{k+l+3/2} \int_0^1 s^k (1 - s)^{l+1/2} ds. \end{aligned}$$

Hence  $g$  satisfies the assumption in Lemma 3.1 with  $0 < n = k + 3/2 < 3 + l$  by the assumption on  $k$ .

### 4.1.2 The relativistic Vlasov-Poisson system

In this case the corresponding function  $g$  is defined in (2.7). We observe that  $(1 + y - \eta)^2 - 1 = (1 + y - \eta + 1)(1 + y - \eta - 1) \geq y - \eta$  and find that

$$\begin{aligned} g(y) &\geq C \int_0^y \eta^k (1 + y - \eta) ((1 + y - \eta)^2 - 1)^{l+1/2} d\eta \\ &\geq C \int_0^y \eta^k (y - \eta)^{l+1/2} d\eta \\ &= Cy^{k+l+3/2} \int_0^1 s^k (1 - s)^{l+1/2} ds. \end{aligned}$$

Again,  $g$  satisfies the assumption in Lemma 3.1.

### 4.1.3 The Einstein-Vlasov system

In this case the corresponding function  $g$  is defined in (2.11). We estimate analogously to 4.1.2, and in addition we observe that for  $y$  sufficiently small,

$1 - e^{-y} \geq y/2$  and  $e^y \geq 1/2$ . Hence

$$\begin{aligned} g(y) &\geq C \int_0^{1-e^{-y}} \eta^k ((1-\eta)^2 - e^{-2y})^{l+1/2} d\eta \\ &\geq C \int_0^{1-e^{-y}} \eta^k (1-\eta - e^{-y})^{l+1/2} d\eta \\ &= C(1 - e^{-y})^{k+l+3/2} \int_0^1 s^k (1-s)^{l+1/2} ds \geq C y^{k+l+3/2} \end{aligned}$$

as desired. We collect the results for the kinetic models into a theorem.

**Theorem 4.1** *Let  $\Phi$  satisfy the assumptions stated in 2.1.1 and (4.1). Then for any  $\mathring{y} > 0$  the reduced equation (2.5) or (2.13) has a unique solution  $y \in C^1([0, \infty[)$  with  $y(0) = \mathring{y}$  which has a unique zero  $R > 0$ . By*

$$f(x, v) = \Phi \left( y(r) - \frac{1}{2}|v|^2 \right) |x \times v|^{2l}$$

or

$$f(x, v) = \Phi \left( 1 + y(r) - \sqrt{1 + |v|^2} \right) |x \times v|^{2l}$$

or

$$f(x, v) = \Phi \left( 1 - e^{-y(r)} \sqrt{1 + |v|^2} \right) |x \times v|^{2l}$$

a static, spherically symmetric solution to the Vlasov-Poisson or relativistic Vlasov-Poisson or Einstein-Vlasov system is defined. This solution is compactly supported, and its spatial support is the ball with radius  $R$  centered at the origin. The parameter  $\mathring{y}$  is related to the potential  $U$  or the metric quantity  $\mu$  via  $\mathring{y} = U(R) - U(0)$  or  $\mathring{y} = \mu(R) - \mu(0)$  respectively. Moreover,  $\rho, p \in C(\mathbb{R}^3) \cap C^1(B_R(0))$ .

## 4.2 Fluid models

Let  $P$  satisfy the assumptions stated in 2.2.1 or 2.2.2. In addition let

$$P'(\rho) \leq c\rho^{1/n} \text{ for } \rho \in ]0, \rho_0[ \quad (4.2)$$

for some parameters  $c > 0$ ,  $\rho_0 > 0$ , and  $0 < n < 3$ . It turns out that in checking the condition for the corresponding function  $g$  we need not distinguish between the non-relativistic and relativistic cases, since in both cases

$$Q(\rho) \leq \int_0^\rho \frac{P'(s)}{s} ds \leq C\rho^{1/n}$$



for  $0 < \rho < \rho_0$ . Since for positive arguments,  $g$  is defined as the inverse function to  $Q$  this immediately yields the estimate for  $g$  required in Lemma 3.1, and we can sum up our results for the fluid case.

**Theorem 4.2** *Let  $P$  satisfy the assumptions stated in 2.2.1 or 2.2.2 and (4.2). Then for any  $\mathring{y} \in ]0, y_{\max}[$  the reduced equation (2.5) or (2.13) has a unique solution  $y \in C^1([0, \infty[)$  with  $y(0) = \mathring{y}$  which has a unique zero  $R > 0$ . By*

$$\rho = g(y), \quad p = P(\rho)$$

*with  $g$  defined in 2.2.1 or 2.2.2 respectively, a static, spherically symmetric solution to the Euler-Poisson or Einstein-Euler system is defined. This solution is supported in the ball of radius  $R$  centered at the origin. The parameter  $\mathring{y}$  is related to the potential  $U$  or the metric quantity  $\mu$  via  $\mathring{y} = U(R) - U(0)$  or  $\mathring{y} = \mu(R) - \mu(0)$  respectively. Moreover,  $\rho \in C(\mathbb{R}^3) \cap C^1(B_R(0))$ .*

### 4.3 Final remarks

1. In the kinetic case it is straight forward to extend the above analysis to an ansatz of the type

$$f(x, v) = \Phi(E_0 - E) (L - L_0)_+^l$$

or its general relativistic analogue, where  $L_0 > 0$  and the other parameters are as before. Such an ansatz leads to steady states which have a vacuum region at the center. This situation was investigated in [19] by a perturbation argument, and the structure of the resulting steady states was studied in [1] by numerical means.

2. The arguments from Lemma 3.1 can easily be applied to show that a solution to the equation (2.4) with data  $y(\mathring{r}) = \mathring{y} > 0$ ,  $y'(\mathring{r}) = \mathring{y}'$  prescribed at some radius  $\mathring{r} > 0$ , has a zero to the right of  $\mathring{r}$ , provided  $\mathring{y}' < 0$ . The important point is that again  $y'(r) \leq -m(r)/r^2$  for  $r \geq \mathring{r}$ , where  $m(r) = 4\pi \int_{\mathring{r}}^r s^{2+2l} \rho(s) ds$ . This extension will be useful in [16].
3. In the Euler case there is a size restriction on  $\mathring{y}$ , if  $y_{\max} < \infty$ , i.e., if  $P$  grows only sub-linearly for large values of  $\rho$ . If the pressure is weak in this sense it cannot support an arbitrarily large potential difference between the center and the surface of the equilibrium matter distribution. For equations of state which typically arise in physics  $P$  grows superlinearly for large values of  $\rho$  so that  $y_{\max} = \infty$ . To see why no such restriction appears in the kinetic case we consider for simplicity

the Vlasov-Poisson case with  $l = 0$ . Then  $p = h(y) = h(g^{-1}(\rho))$ , i.e.,  $P = h \circ g^{-1}$ . A simple change of variables shows that in this case

$$Q(\rho) = \int_0^\rho \frac{(h \circ g^{-1})'(s)}{s} ds = \int_0^{g^{-1}(\rho)} \frac{h'(t)}{g(t)} dt \rightarrow \infty \text{ as } \rho \rightarrow \infty$$

because in the isotropic Vlasov case  $h'$  is a positive multiple of  $g$ , cf. [25, Lemma 2.2]. Hence in the Vlasov case  $y_{\max} = \infty$ .

4. In the kinetic case,  $\rho \in C^1(\mathbb{R}^3)$ , provided  $\kappa + l + 1/2 > 0$ , and the same is true in the fluid case under a suitable assumption on  $P$ .
5. In the kinetic case the restriction  $l > -1/2$  can be relaxed by assuming more on  $\Phi$ . For example, Lemma 3.1 applies to all the polytropes (1.22) in the Vlasov-Poisson case with  $k, l > -1$ ,  $k + l + 3/2 > 0$  and  $k < l + 3/2$ , since in that case  $\rho(r) = cr^{2l}y(r)_+^{k+l+3/2}$ .

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## References

- [1] ANDRÉASSON, H., REIN, G., On the steady states of the spherically symmetric Einstein-Vlasov system. *Class. Quantum Grav.* **24**, 1809–1832 (2007).
- [2] BATT, J., FALTENBACHER, W., HORST, E., Stationary spherically symmetric models in stellar dynamics. *Arch. Rational Mech. Anal.* **93**, 159–183 (1986).
- [3] BINNEY, J., TREMAINE, S., *Galactic Dynamics*, Princeton University Press, Princeton 1987.
- [4] CHANDRASEKHAR, S., *An introduction to the study of stellar structure*, Dover Publications, New York 1958.
- [5] FJÄLLBORG, M., HEINZLE, J., UGGLA, C., Self-gravitating stationary spherically symmetric systems in relativistic galactic dynamics. *Math. Proc. Camb. Phil. Soc.* **143**, 731–752 (2007).
- [6] GIDAS, B., NI, W.-M., NIRENBERG, L., Symmetry and related properties via the maximum principle. *Commun. Math. Phys.* **68**, 209–243 (1979).

- [7] GUO, Y., REIN, G., A non-variational approach to nonlinear stability in stellar dynamics applied to the King model. *Commun. Math. Phys.* **271**, 489–509 (2007).
- [8] HADŽIĆ, M., REIN, G., Global existence and nonlinear stability for the relativistic Vlasov-Poisson system in the gravitational case. *Indiana University Math. J.* **56**, 2453–2488 (2007).
- [9] HEINZLE, J., RENDALL, A., UGGLA, C., Theory of Newtonian self-gravitating stationary spherically symmetric systems. *Math. Proc. Camb. Phil. Soc.* **140**, 177–192 (2006).
- [10] HEINZLE, J., RÖHR, N., UGGLA, C., Dynamical systems approach to relativistic spherically symmetric static perfect fluid models. *Class. Quantum Grav.* **20**, 4567–4586 (2003).
- [11] HEINZLE, J., UGGLA, C., Newtonian stellar models. *Ann. Phys. (NY)* **308**, 18–61 (2003).
- [12] JANG, J., Nonlinear instability in gravitational Euler-Poisson system for  $\gamma = 6/5$ . *Arch. Rational Mech. Anal.* **188**, 265–307 (2008).
- [13] LEMOU, M., MÉHATS, F., RAPHAËL, P., Orbital stability of spherical galactic models. *Invent. math.* **187**, 145–194 (2012).
- [14] MAKINO, T., On spherically symmetric stellar models in general relativity. *J. of Math. of Kyoto University* **38**, 55–69 (1998).
- [15] RAMMING, T., *Über Familien sphärisch symmetrischer stationärer Lösungen des Vlasov-Poisson-Systems*, PhD thesis, Bayreuth 2012.
- [16] RAMMING, T., Mass-radius diagrams for families of steady states of the Vlasov-Poisson system. *In preparation*.
- [17] REIN, G., Static solutions of the spherically symmetric Vlasov-Einstein system. *Math. Proc. Camb. Phil. Soc.* **115**, 559–570 (1994).
- [18] REIN, G., *The Vlasov-Einstein System with Surface Symmetry*, Habilitationsschrift, München 1995.
- [19] REIN, G., Static shells for the Vlasov-Poisson and Vlasov-Einstein systems. *Indiana University Math. J.* **48**, 335–346 (1999).
- [20] REIN, G., Reduction and a concentration-compactness principle for energy-Casimir functionals. *SIAM J. Math. Anal.* **33**, 896–912 (2002).

- [21] REIN, G., Non-linear stability of gaseous stars. *Arch. Rational Mech. Anal.* **168**, 115–130 (2003).
- [22] REIN, G., Collisionless kinetic equations from astrophysics—The Vlasov-Poisson system. In *Handbook of Differential Equations, Evolutionary Equations, vol. 3*, ed. by C. M. Dafermos and E. Feireisl, Elsevier (2007).
- [23] REIN, G., RENDALL, A., Global existence of solutions of the spherically symmetric Vlasov-Einstein system with small initial data. *Commun. Math. Phys.* **150**, 561–583 (1992).
- [24] REIN, G., RENDALL, A., Smooth static solutions of the spherically symmetric Vlasov-Einstein system. *Ann. de l'Inst. H. Poincaré, Physique Théorique* **59**, 383–397 (1993).
- [25] REIN, G., RENDALL, A., Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics. *Math. Proc. Camb. Phil. Soc.* **128**, 363–380 (2000).
- [26] RENDALL, A., B. SCHMIDT, B., Existence and properties of spherically symmetric static fluid bodies with a given equation of state. *Class. Quantum Grav.* **8**, 985–1000 (1991).
- [27] SANSONE, G., Sulle soluzioni di Emden dell'equazione di Fowler. *Rend. Mat. Roma* **1**, 163–176 (1940).